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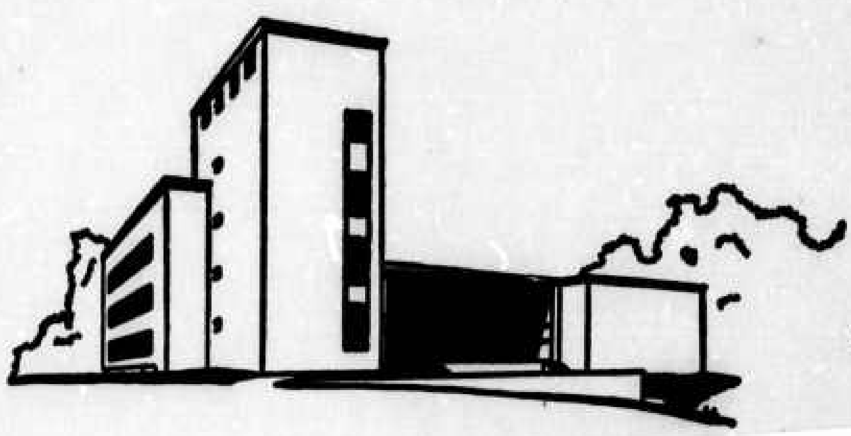

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C^2 AND LPU^2 COMBINATIONS FOR TREATING DIFFERENT RISKS AND UNCERTAINTIES IN CAPITAL BUDGETS

by

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ABSTRACT

Chance Constrained (C^2) Programming and Linear Programming under Uncertainty (LPU^2) are joined together in order to deal with different risks and uncertainties which are commonly encountered in capital budgeting. This includes payback period protection via chance constraints formulated to cover (or bound) a possible loss of future opportunities during the payback period. It also includes liquidity requirements formulated preemptively via LPU^2 to provide protection against possible cash (or liquidity) shortages at specified times.

The case of arbitrary discrete distributions is examined and new formulations are developed which model economic, statistical, and technological decision interdependencies. Relations to geometric programming are indicated prior to reducing these formulations to zero-one integer programming (deterministic) equivalents. Duality relations obtained from these formulations provide separate evaluators for yield, risk, portfolio and liquidity effects of cash investment. Finally, relations to "Balas-type" subsidy and penalty adjustments are noted.

I. Introduction

This paper is a sequel to earlier ones [4,5] [1] which detailed some new approaches to capital budgeting under risk. We may here recall that these previous papers introduced a joint use of chance constraints (Chance-Constrained, or C^2, Programming) and Linear Programming Under Uncertainty (LPUU) in order to provide new ways of dealing with risks in the different dimensions that are likely to be encountered in modeling for realistic capital budgeting. The objective was to open new avenues for an operational approach to such multi-dimensional situations of risk which may be encountered in actual applications, and, simultaneously, to achieve a possibly better understanding of common practices with regards to payback-liquidity. To facilitate understanding, only the simplest types of situations were considered -- e.g., only payback and liquidity were considered to represent some of the different dimensions of risk and only the class of zero-order rules in C^2 Programming were treated in explicit detail. The present paper is also confined to these as the simplest situations which can lend themselves to the kinds of understanding which are wanted at this juncture.

When the probability distributions of cash flows are continuous there are various problems of computation that remain to be resolved, especially in those cases for which the distributions of cash flows are not normal (or log-normal), since in other cases the precise nature of the deterministic equivalents, if any, have still to be detailed when the kinds of constraints we shall consider are assumed to involve arbitrary statistical distributions or classes of decision rules. In some of these cases one can utilize various devices of approximation, etc., -- e.g., non-normal distributions might be approximated by convex combinations of normal distributions--and, of course, various types of transformations and reductions can also be employed. [2]

1/ See also [6]

2/ See e.g., [12]

However, size and computational effort will generally increase rapidly, and theoretical interpretation of results becomes difficult, as well.

In the light of difficulties like these, it has seemed worthwhile to investigate other possible approaches and, in particular, it has seemed worthwhile to turn to discrete distributions.3/ Such distributions had not been explored previously in C^2 programs,4/ although, of course, they are of interest in their own right as well as for approximating continuous distributions.5/ Chance constraints which employ discrete distributions are especially attractive for capital budgeting since these are the ones most likely to be available insofar as any data on frequencies are available at all.

We shall naturally be concerned with arbitrary discrete distributions. Furthermore, we shall show how one may utilize multivariable distributions of this form together with combinations of decision variables in ways which can accommodate technological and economic interdependencies (substitutability, complementarity, etc.) as well as stochastic relations that may be of interest in capital budgeting.6/

3/ See e.g., [6] and [3]

4/ Our attention has been called to certain work [23] by W. Raike in this area.

5/ See [7]

6/ See [6] for a detailed development. See also Hillier [15] for comments and suggestions in a similar vein.

To obtain the interpretations which are of interest, these probabilistic formulations will be reduced to certain nonlinear deterministic equivalents[7/] and then the latter will, in turn, be transformed into a 0-1 integer (linear) programming equivalent. Within the text of this article this integer programming equivalent will be interpreted somewhat loosely as an ordinary linear program and related to the work of Weingartner [31] and others. Accompanying remarks will provide needed qualifications and a tie-in to an appendix that relates this all to the exact duality characterizations provided by E. Balas [1] for 0-1 programming.

We can now best conclude this introduction by noting that, for the most part, this paper is a summary which attempts to render previous results in a more compact and sharper form. Conversely, the reader interested in more detail may refer to [3] and [6] in the bibliography that is appended to this paper.

7/ These can also be interpreted in terms of geometric programming. Cf. [6] where this is done and where the solutions and interpretations are also treated via the convex-approximant procedures developed in [8].

2. Discrete Distributions of Cash Flows

To deal with capital budgeting when the cash flows associated with projects are described by discrete distributions, we introduce some definitions and make some simplifications, as follows. Let

x_{ij} = the fraction of project i adopted in period j

d_i^k = chance variable for the net cash flow k periods after the start of project i

(1) d_{si}^k = the net cash flow at level s (i.e., the scalar value associated with the $s^{\underline{th}}$ level of cash flow) from project i, k periods after the start of the project.

p_{si}^k = the probability of occurrence of d_{si}^k

T = a prescribed payback period

α = a minimum probability of payback--e.g., as specified by management

d_i^o = the investment required initially for project i, assumed known with certainty,

"E" = represents the expected value operator.

i = 1,...., I, where I is the number of projects under consideration

j = 1,...., J, where J is the number of decision periods under consideration.

Figure 1 is illustrative of a set of such cash flow estimates as they might appear for an hypothetical project. Note that the sample space we are dealing with is finite--i.e., these spaces have only a finite number of points--and hence the probabilities may be dealt with directly. No mediation via density functions and corresponding integral expressions is required.

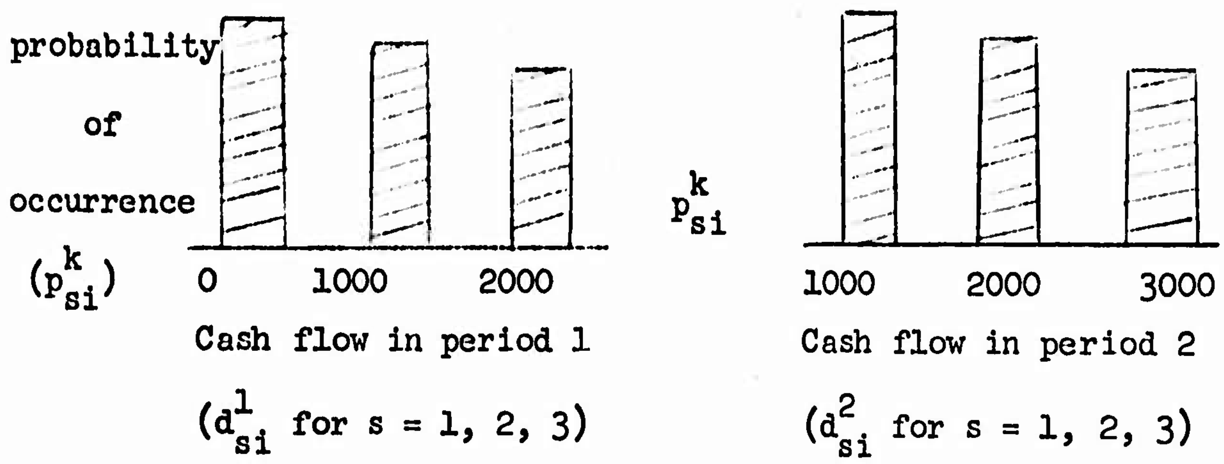

Figure 1 - Discrete Probability Distributions of Cash Flows for an Hypothetical Project

With discrete probabilities, the expected value of cash flow for a single project in the kth period after adoption is $\sum_s p_{si}^k \; d_{si}^k$. With multiple independent projects the expected value of cash flow in period k from all projects adopted in decision period j is simply the sum of the expected values for each project adopted, which we may represent as $\sum_i \sum_s p_{si}^k \; d_{si}^k \; x_{ij}$ -- by recourse to the notation exhibited in (1) above. Acceptance of project i in decision period j occurs only if $x_{ij} = 1$, which adds to this summation all of the payoff terms associated with project i. Conversely rejection of project i in period j produces $x_{ij} = 0$ and the related terms are then excluded from the summation.

We shall, as before, continue with an expected value objective subject to payback and related risk-control constraints.[8/] In the case of discrete distributions, there is a simplification in that the total probability of a payoff exceeding a specified level is found by directly summing the discrete probabilities of occurrence for all those outcomes which exceed the specified level. For example, considering Figure 1, the probability of the cash flow in period 1 exceeding 500 is simply the sum of the probability of a cash flow of 1000 and the probability of a cash flow of 2000. The probability of a cash flow in period 2 exceeding 500 is the sum of the probabilities of flows of 1000, 2000, and 3000. With discrete distributions each cash flow level represents a

8/ As previously explained in [3] and [4] we regard such a chance-constrained programming formulation for payback as representing a bound for dealing with uncertainties arising from the possibility of better profit opportunities materializing with some unknown probabilities during some prescribed time horizon.

probabilistic event. The probability of occurrence of any particular combination of outcomes from a number of projects and/or over a number of periods can be found according to the rules that usually govern the computation of combinations of probabilistic events. In particular, the multiplication rule which relates the joint, discrete, and conditional probabilities of two events A and B, say, may be utilized in the form

$$P(A \cap B) = P(B)\ P(A/B) = P(B/A)\ P(A)$$

so that no particular assumptions need then be made with respect to events which have zero probability of occurrence. This multiplication rule applies whether or not the events are independent.

An application of the model to interdependent events requires only a determination of the relevant conditional distributions. Also, we shall arrange our model, relative to the decision variables, in a way that allows additional flexibility in dealing with either statistical interdependence, economic interdependence, or both.9/ E.g., if we are considering two projects, the probability of any particular pair of payoffs occurring in period k will be expressed as10/ $p_{s1}^k\ p_{s2}^k\ x_1\ x_2$ where $x_i = 1$ if a project is adopted, $x_i = 0$ if it is rejected. Thus this expression will be equal to 0 unless <u>both</u> project 1 <u>and</u> project 2 are adopted ($x_1 = 1$ <u>and</u> $x_2 = 1$). Similarly, the probability of occurrence of specific cash flows from any group of projects can be expressed by product terms which can reflect a decision interdependence to any desired degree and which take on the appropriate probability value if all the projects are adopted and are zero otherwise.

Now consider a payback-period requirement formulated as a chance constraint. A deterministic equivalent to such a payback constraint can be obtained in principle by first enumerating all possible combinations of cash flows during

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9/ A development may be found in [6] which relates the former (economic-technological interdependencies such as complementarity, etc.) to the latter (statistical interdependencies including portfolio risks, etc.).

10/ To simplify notation we are omitting the subscript j associated with the decision period for these variables.

the required payback period, T. Consider the case of only one project, for instance, so that we can omit the subscript i in our d^k_{si}. Those cash flows for which the sum of the values d^k_s for the random variables is equal to or greater than zero--the outcomes, if any, which satisfy $\sum_{k=0} d^k_s \geq 0$--are the set which meet the payback constraint. Since we have a probability associated with each of these d^k_s values the product of these probabilities will be the probability of a particular outcome.

The process can be represented as a probability tree, where each final branch represents a particular combination of period cash flows and the probability of achieving that combination.

For notational convenience, we define the quantity

D_s = the sum of a possible series of cash flows (d^k_s) over a specified period T. For example, each unique path through the three stage probability tree (Figure 2) represents one value of D_s, e.g., for T = 3. This is the value determined by summing the values of d^k_s from a mutually exclusive set of outcomes. Having enumerated the set of values for D_s we can determine the probability of occurrence of each value as the product of the probabilities of occurrence associated with each branch.

We will designate the probability of a specific outcome D_s as Pr_s. As we shall see this identification is in the nature of an isomorphism which permits us to replace the constraints on the d^k_s with constraints on the probabilities only, relative to α . The set of possible outcomes is a finite set which we have indexed over s, and we may partition this set into those s for which D_s equals or exceeds a specified level and those for which it does not. E.g., we may specify the set

$$S = \left\{ s: \quad D_s \geq 0 \right\}$$

as the set of cash flows over T periods which is non-negative (i.e., meets the payback requirement).

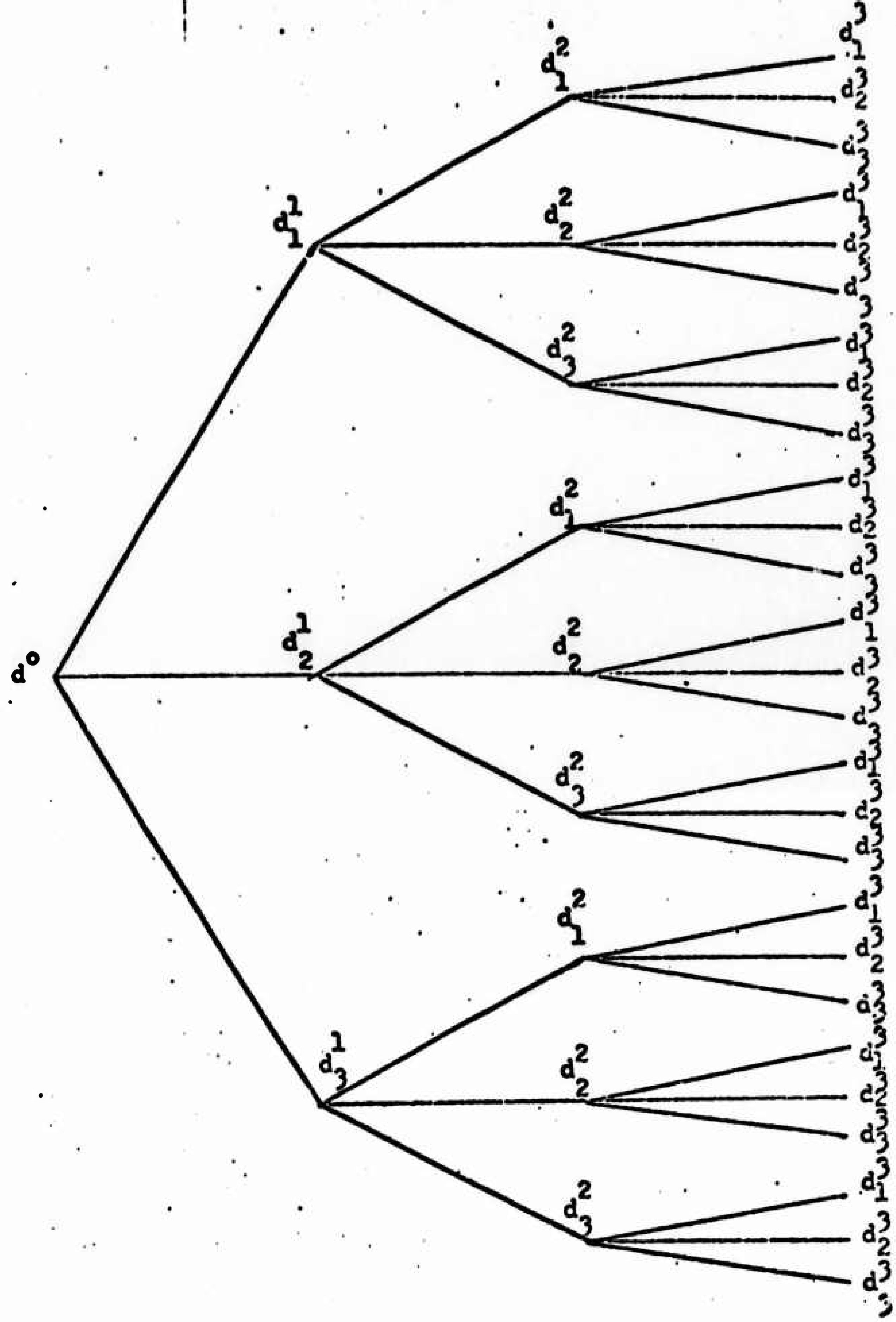

Figure 2

A Three Stage Cash Flow Tree

For any set of D_s we determine the probability of achieving some specified value by summing the Pr_s associated with the set of D_s which achieve the specified value. Therefore:

$\sum_{s \in S} Pr_s$ = the probability of achieving payback if the project is adopted.

We now turn to ways for simplifying the analysis by observing that the final branches of the tree, that is, the total set of D_s values, can be divided into two exhaustive, mutually exclusive sets - namely, those outcomes which meet the payback constraint ($D_s \geq 0$) and those which do not ($D_s < 0$). As a direct consequence of this, we can write the payback constraint in two equivalent forms. E.g., we may write

$$\sum_{s \in S} Pr_s \, x \geq \alpha$$

which represents the requirement that the probability of achieving payback must equal or exceed α , where $0 \leq \alpha \leq 1$ is a prescribed measure of risk or probability and x, a scalar, represents the decision variable which is applicable to this one-project case. Alternatively we may write

$$\sum_{s \in \tilde{S}} Pr_s x \leq (1 - \alpha) \leq 1.$$

(2)

which is the complementary requirement, viz., the probability of not achieving payback must not exceed $0 \leq (1 - \alpha) \leq 1$.

We now define this probability for $D_s < 0$ as:

$P = \sum_{s \in \tilde{S}} Pr_s$ = the probability of not achieving payback if the project is adopted.

In any event we have achieved a simplification for the case of discrete distributions in that we now need to deal only with this one linear constraint and thereby eliminate the double inequality (nonlinearity) that is a part of the chance constraint in the case of continuous distributions (See [3]).[11]

We may now formally proceed to write

(3)
$$\text{Maximize } Z = \sum_{k=0}^{J} \sum_{s} p_s^k \, d_s^k \, x$$

Subject to

(a) $P\,x \leq 1 - \alpha$

(b) $0 \leq x \leq 1$, x an integer

as an ordinary 0-1 integer programming problem for this one-project case. This model is deterministic but, of course, it is too simple to bear the weight of further extensions to multiple-projects and other types of risk constraints. On the other hand, we shall soon see that even these more complex cases can also be reduced to 0-1 deterministic equivalents and have the structure of this one-project problem.

11/ Although we have, for the sake of simplicity, developed this example by assuming that the cash flows in any period are statistically independent of those in other periods, this involves no real loss of generality, at least in principle, since the method can be extended to statistically interdependent distributions where cash flows are dependent on the cash flow levels occurring in prior years. Such an extension requires only that the relevant distributions be estimated and hence adds no conceptual or representational problems, although, of course, it does require extensions of the computations by such prior analyses and reductions as may be necessary to secure the p_{si}^k values for each interdependency between the d_{si}^k.

If cash flows are dependent on cash flows in prior years, then the relevant probability distributions to use for cash flows in periods $k = 2, 3, \ldots$, etc. are the conditional distributions of cash flows. This means that in each period there may be as many distributions to estimate as there are discrete levels of cash flow in the previous period. For instance in the tree diagram (Figure 2), the distribution of cash flow during the second period (d^2) would have to be estimated separately for $d^1 = d_1^1$, for $d^1 = d_2^1$, and for $d^1 = d_3^1$. This would not change the number of possible outcomes (the size of the total set of D_s or branches on the tree, nor the computation of D_s' and P. The probability of a particular outcome, D_s, is still found by multiplying the probabilities of the d_{si}^k of which it is comprised.

3. Extension to Multiple Projects

Having examined the single project case and some related interpretations in a very simple context, we now proceed to more complex cases. In particular we extend the model to multiple projects, while allowing for cross effects between projects in a variety of ways.

At this point it is well to reiterate the general setting in which we are exploring the problem of capital budgeting under risk. The firm is faced with a total of I possible investment projects, identified by the subscripts $i = 1, \ldots, I$. It has made estimates of the cash flows associated with each of these projects, and has described these cash flows in the form of discrete distributions. It wishes to plan its investment program over a specified horizon of J years, and will commit itself <u>now</u> to the projects it will initiate during each of these $j = 1, \ldots, J$ years (i.e., we are supposing that a zero-order decision rule applies),[12/]

In order to protect against the <u>uncertainty</u> of possibly "even better" investment opportunities arising, it is assumed that a <u>risk</u> reduction is made via a payback period constraint or policy. In particular it is assumed that the applicable policy is formulated so that the <u>portfolio</u> of investments undertaken in any year must have a probability α of reaching a total non-negative cash flow within T years. This will hereafter be referred to as a portfolio payback requirement and the related "opportunity risk" coverage it provides will be distinguished from constraints formulated to provide protection against

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12/ Cf. [9].

still other types of risk. (For instance a constraint designed to maintain a specified minimum level of liquidity at prescribed points in time will also be utilized in the form of a "solvency requirement" which is to be met with probability 1 at the end of each period.)[13/]

We now develop the probability expressions needed to extend (3) for the deterministic equivalents of this more general problem. The probability of any particular series of cash flows occurring during the T years following the initiation of a specific project is $\prod_{k=0}^{T} p_{si}^{k}$, the product of the probabilities of each of the individual cash flows which comprise the series. By summing the probabilities of each of those mutually exclusive series which achieves the specified outcome (i.e., for non-payback those which fall in the range $D_s < 0$) we obtain the probability of achieving this outcome if any one project is adopted.

We shall hereafter assign a subscript to P when we want to indicate its association with a specific project. Clearly, we m[illegible] than apply the method presented in section 2 to any number of individual projects, thereby determining the appropriate probability of non-payback (P_i) for each.

If more than one project is adopted, however, the joint probabilities must be considered in case either a) projects are statistically interdependent, b) the payback requirement is to be applied on a portfolio basis, or c) both these conditions apply. Recalling that b) follows from our interpretation of payback as a risk constraint designed to hedge (via an inequality) for the possibility of lost opportunities, it then also follows that we should undertake consideration of the joint interdependent probabilities.

13/ Still other types of risk may involve a use of posture constraints either at prescribed points in time or at the end of a planning horizon. Such horizon posture constraints, we should note, can be especially useful in establishing inequality bounds on beyond-the-horizon opportunities (or requirements) by reference to the full detail that is available from within-the-horizon components (including the dual evaluators) of a model. Thus, here again, an alternative (or complement) for the use of a single summary discount rate is available to take account of beyond-the-horizon considerations.

We proceed to further details as follows. The definition of D_s is extended to include the outcomes resulting from adoption of more than a single project. Observe that (d^1_{1A}), the cash flow at level d^1_1 for project A, may occur in combination with a cash flow level of d^1_1, d^1_2, or d^1_3 from project B. We can, for each pair of projects, again determine the set of all possible outcomes which achieve a specified level, and express this as:

$$S_{AB} = \left\{ s: \quad D_s < 0 \right\}$$

where the subscript identifies the pair of projects which define the set S_{AB} that is of interest. We establish the non-payback probability of this set by taking the product of the probabilities of each possible pair of cash flows in the initial period times the product of the probabilities of each possible pair of cash flows in each succeeding period through T. Then we sum these probabilities over all the final outcomes for which the total cash flow achieves the specified level (i.e., for non-payback we sum over all outcomes for which total cash flow is negative). If we then multiply this probability by x_A x_B, we have an expression for the probability of not achieving payback from a program consisting of only projects A and B. Observe, however, that when either $x_A = 0$ or $x_B = 0$, these joint probability expressions take on a value of zero. It takes on the appropriate probability value only when <u>both</u> projects are adopted (i.e., when $x_A = 1$ <u>and</u> $x_B = 1$).

Recall that we want to consider the entire distributions of relevant probabilities. Hence it is not sufficient to stop with this two-project case as we could do if we used only variance-covariance (or semi-variance)[14] measures of risk. To be able to consider any finite number of joint-project possibilities, we therefore proceed as follows. To obtain a function whose value is the probability of achieving a specified outcome, we further sum a series consisting of the probabilities associated 1) with each individual project, 2) with each pair of projects, 3) with each triad, etc.

14/ See, e.g., H. M. Markowitz [20].

This is all that needs to be said in principle since the general path of development is now evident - - viz., as in the two-project case, associating an x_i with each term, as appropriate, assigns a value of zero to any term including a rejected project, etc. Per contra, each term which includes only accepted projects ($x_i = 1$) will have a non-zero value, so that we thereby allow for the statistical (and other) dependencies that are appropriate via this mode for model development. Appendix A details the development of the appropriate probability expressions.

4. Linear Programming Models Under Certainty

At this point we may put these preliminary developments to work in a way that will help to relate this to some of the work for certainty models as previously developed by Weingartner in [31] in order to advance the state-of-the-art of capital budgeting theory and practice. This should help to illuminate a train of developments that relates back even to [26.1] as well as subsequent developments in financial-budgetary planning.

Thus in these developments we shall follow Weingartner's initial developments and avoid dealing with any integer requirements in our interpretive discussions. The latter will then be elaborated subsequently in an appendix.

In moving from certainty models of capital budgeting we want to show how a risk extension modifies such a certainty model and, in particular, how it modifies the criteria by which an acceptable project may be defined. In this way we shall see how the presence of risk imparts some new interpretations of the capital budgeting problem. For instance one analysis of these portfolio risks will help us to point up the possible inadequacies of approaches which propose the application of a "risk adjustment factor" (e.g., the application of a higher discount rate) to individual projects. Other interpretations will be used to distinguish between yields and risks on individual investments and their related portfolio effects as well. But, in any event, it will be well for us to use the work of Weingartner as a framework in order to relate these materials to the preceding work that he has succeeded in incorporating in his models.

We turn to one of Weingartner's "final" models [15/] in which funds may be borrowed but at increasing rates of interest as the amount of borrowing increases. We shall assume that the firm may invest any surplus funds it wishes at some rate of interest equal to or less than the minimum borrowing rate. For simplification, it is assumed that borrowing and lending are accomplished by means of "renewable" one year contracts. The objective is to maximize the value of the firm's assets as of the planning horizon, as represented by the value of physical assets and cash. The funds available to support the investments are determined by the cash throw-off of the firm's existing resources each year.

In order to present this model, we let

D_j = the funds anticipated to be generated by the firm from operations in year j of the resources the firm currently controls; or equivalently anticipated throw-off based on continuation of the firm but excluding revenues to be derived from investments which the model is designed to determine,

r_n = the interest rate applicable to the n^{th} step of the (marginal)

(4) supply curve,

w_{nj} = the amount borrowed in this n^{th} step in year j,

K_n = the upper limit of the n^{th} step.

$\hat{d}_i$ = the value of all flows generated by an investment subsequent to the horizon [16/] discounted at an appropriate rate of interest.

v_j = the amount available for lending in year j,

r_L = the rate of interest available for investment funds,

15/ See [31], pp. 168 ff.

16/ This follows the development provided by Weingartner rather than the horizon posture constraint alternative that was footnoted on page 10.

Then, utilizing these definitions together with those of (1), this version of the "Weingartner Capital Budgeting Model" becomes

$$\text{Maximize} \quad \sum_{i=1}^{I} \hat{d}_i x_i + v_J - \sum_{n=1}^{N} w_{nJ}$$

Subject to

$$\text{(5)} \quad \begin{array}{lll} \text{(a)} & \sum_{i=1}^{I} (-d_i^1 x_i) + v_1 - \sum_{n=1}^{N} w_{n1} \leq D_1 & \\ \text{(b)} & \sum_{i=1}^{I} (-d_i^j x_i) - (1 + r_L) v_{j-1} + v_j + \sum_{n=1}^{N} (1 + r_n) w_{n,j-1} + & \\ & \quad + \sum_{n=1}^{N} w_{nj} \leq D_j, & j = 2,\dots,J \\ \text{(c)} & w_{nj} \leq K_n, & j = 1,\dots,J;\ n = 1,\dots,N \\ \text{(d)} & 0 \leq x_i \leq 1, & i = 1,\dots,I \\ \text{(e)} & v_j,\ w_{nj} \geq 0 & j = 1,\dots,J;\ n = 1,\dots,N \end{array}$$

Constraint (5a) requires that the net cash outflow of the project from its first year of operation plus surplus cash at the end of the year, less the amount borrowed, must not exceed the cash throw-off from present operations during the year.[17/] Constraints (5b) apply a similar requirement to the subsequent years within the horizon period, adding the interest earned from lending or paid for borrowings.

The rates of interest are associated with steps in a marginal cost of funds schedule and are ordered so that $r_{n-1} < r_n < r_{n+1}$. This relation between the rates of interest on adjacent steps eliminates the necessity for stating lower limits, since the alternative to borrowing at a rate r_m is borrowing at a lower, hence preferred, rate r_s with $s < m$. If borrowing at rate r_m takes place it is because the limited amount available at lower rates has been exhausted.

17/ To simplify notation we have collapsed the initial investment d_i^0 and the cash flow generated during the first year into a single value, d_i^1.

The dual problem to (5) is evidently

$$\text{Minimize } \sum_{j=1}^{J} \lambda_j D_j + \sum_{i=1}^{I} \mu_i + \sum_n \sum_j K_{nj} K_n$$

$$\text{Subject to (a) } \sum_{i=1}^{I} -\lambda_j d_i^j + \mu_i \geq \hat{d}_i. \qquad i = 1,\ldots,I$$

$$\text{(b) } \lambda_J \geq 1$$

$$\text{(6)} \qquad \text{(c) } -\lambda_J + K_{nJ} \geq -1, \qquad n = 1,\ldots,N$$

$$\text{(d) } \lambda_j - (1 + r_L) \lambda_{j+1} \geq 0 \qquad j = 1,\ldots,J-1$$

$$\text{(e) } -\lambda_j + (1 + r_n) \lambda_{j+1} + K_{nj} \geq 0 \quad j = 1,\ldots,J-1 \qquad n = 1,\ldots,N$$

$$\text{(f) } \lambda_j, \mu_j, K_{nj} \geq 0$$

For any accepted project (i.e., $x_i > 0$), (6a) will hold as an equality as a condition of optimality. Thus, still following Weingartner, we obtain

$$\overset{*}{\mu}_i = \hat{d}_i + \sum_{j=1}^{J} \overset{*}{\lambda}_j d_i^j$$

From this we see that for accepted projects $\overset{*}{\mu}_i \geq 0$ is simply the discounted sum of cash flows subsequent to the horizon plus those up to the horizon associated with a particular project, i. The discount rates, $\overset{*}{\lambda}_j$, are compound interest rates for (J-j) periods at the appropriate period rates. As Weingartner shows,[18] the appropriate period rates are the marginal rates incurred (or earned) during each period. The criterion for project acceptance (when projects are technologically independent) is that the discounted value of the project as of the horizon is equal to or greater than zero, i.e.

$$\hat{d}_i + \sum_{j=1}^{J} \overset{*}{\lambda}_j d_i^j \geq 0$$

18/ Ibid., pp. 171-172.

5. The Extension to Uncertainty

With these developments and interpretations in mind, we next extend the Weingartner model. First we regard the cash flows as stochastic in nature so that the amounts of borrowing and lending must also be regarded as stochastic rather than deterministic variables when the plans are being formulated over the indicated finite horizon. We also add a payback constraint on the projects selected, in order to constrain the possibility of missed opportunities to a desired (inequality) risk level. We still wish to maximize the value of assets as of the horizon year, but since this, too, is a random variable, we can select from a variety of possible objectives[19/] and try to maximize its expected value. This model is then similar to the general model we presented in [4].

The payback constraint we shall consider can then be formulated as[20/]

$$\text{(7)} \qquad \Pr \left(\sum_{k=0}^{T} \sum_{i=1}^{I} d_i^k x_i < 0 \right) \leq 1 - \alpha .$$

Utilizing the developments in Appendix A we may write the deterministic equivalent to (7) as

$$\text{(8)} \qquad \begin{aligned} &\sum_{i=1}^{I} P_i x_i + \sum_{i=1}^{I-1} \sum_{\ell=i+1}^{I} Q_{i\ell} x_i x_\ell + \\ &\sum_{i=1}^{I-2} \sum_{\ell=i+1}^{I-1} \sum_{m=i+2}^{I} Q_{i\ell m} x_i x_\ell x_m + \\ &\sum_{i=1}^{I-3} \sum_{\ell=i+1}^{I-2} \sum_{m=i+2}^{I-1} \sum_{n=i+3}^{I} Q_{i\ell mn} x_i x_\ell x_m x_n \\ &+ \qquad + \quad Q_{123\ldots I} x_1 x_2 x_3 \ldots x_I \leq 1 - \alpha \end{aligned}$$

where P and Q identify non-payback probability expressions as defined in Appendix A.

19/ See e.g. [9]

20/ Other payback constraints (e.g. in bank type mortgage ammortization schedules) could also be utilized to handle partial (interim) payments. See [4]

To reduce the above constraint to an integer linear equivalent we proceed as follows.[20a/] The set of all possible combinations of projects is a finite set which we index over h. M [h] identifies one of these (2^I) combinations containing exactly m_h projects. We then replace each product $\prod x_i$ of <u>two or more</u> decision variables by a single variable y_h defined by

$$y_h = \prod_{i \in M[h]} x_i$$

where we also require the simultaneous satisfaction of

$$(9) \qquad \sum_{i \in M[h]} x_i \leq (m_h - 1) + y_h ,$$

and

$$(10) \qquad \sum_{i \in M[h]} x_i \geq m_h y_h ,$$

with all variables required to be 0, 1 integers.

To see that these new constraints in the variables y_h produce an enlarged problem which is equivalent to the original problem, we first observe that for each <u>product</u> expression in the original problem we have introduced such a y_h as a new variable. But now consider the constraints which relate these y_h values to the original x_i variables. If all the x_i in a specific product expression in the original problem have a value of 1 at the solution, then the corresponding y_h must have a value of at least 1, since under these conditions

$$\sum_{i \in M[h]} x_i = m_h$$

which, with (9), gives

$$m_h \leq m_h - 1 + y_h$$

or $$1 \leq y_h$$

On the other hand, we cannot have $y_h > 1$ since, with all $x_{ij} = 1$, the constraints (10) give

$$m_h \geq m_h y_h$$

or $$1 \geq y_h$$

20a/ I.e., we proceed as in [6] and [3].

Thus $x_i = 1$, all i, implies $y_h = 1$, and conversely, when the x_i are restricted to be at values of 0 or 1 only. Similarly, if any of the x_i in a specific term have a value of 0 at the solution, then the corresponding y_h must be 0, since in this case

$$\sum_{i \in M[h]} x_i < m_h$$

which with (10) requires that

$$1 > y_h$$

But then since y_h is restricted to non-negative integer values, it follows that this last condition implies $y_h = 0$.

Applying this transformation to constraint (8), we replace each product expression containing more than a single decision variable by a new variable y_h and adjoin two additional constraints for each such unique product expression. Thus we obtain for (8)

(11)

$$\text{(a)} \quad \sum_{i=1}^{I} P_i x_i + \sum_{h} Q_h y_h \leq (1 - \alpha)$$

$$\text{(b)} \quad \sum_{i \in M[h]} x_i - y_h \leq m_h - 1$$

$$\text{(c)} \quad -\sum_{i \in M[h]} x_i + m_h y_h \leq 0$$

$$\text{(d)} \quad x_i, y_h \text{ are integers}$$

For verbal compactness, the transformed version of the original constraint (11a) will be identified as the original constraint and the two sets of additional constraints (11c,11d) will be referred to as adjoined constraints. The unity upper bounds are always understood as being involved in the original constraint set, and these are never altered unless otherwise noted. 21/

21/ For clarity we index the constraint coefficients associated with product terms (Q_I) over h = 1,...,H. H has a maximum value of $2^I - I - 1$, consisting of $C_2^I \left(= \frac{I(I-1)}{2}\right)$ second order terms, $C_3^I \left(= \frac{I(I-1)(I-2)}{6}\right)$ third order terms, etc. to a single I^{th} order term C_I^I. This maximum value is attained only when all possible combinations of the I decision variables appear as products in the constraint.

A modification of the Weingartner Capital Budgeting Model can make a closer contact with our own earlier model [4] by utilizing the definitional nature of the lending and borrowing constraints (5a and 5b) to transfer both conditions into the functional. For the deterministic case this will reduce the set of constraints and hence has some virtue, perhaps, even in the deterministic case. For the case of liquidity risk it also allows us to utilize an LPUU formulation that captures the essentials of this kind of risk dimension without having to forego our previous C^2 Programming formulation of opportunity risks as a way of bounding future uncertainties in these dimensions via a risk inequality.

Under the rules of LPUU, as formulated by Dantzig [12], it is assumed that a multi-stage process is available under which one may first observe the outcomes of the various d_i^j and D_j in each period, and then, afterward, one can choose values of v_j or w_h which satisfy (5a) or (5b) as an equality. Thus, we now consider constraint (5a) with the d_i^1 and D_1 as stochastic variables. Then we rewrite this constraint as,

$$v_1 - \sum_{n=1}^{N} w_{n1} \leq \sum_{i=1}^{I} d_i^1 x_i + D_1$$

Since either v_1 or $\sum_n w_{n1}$ (or both) will be zero, at an optimum, we may utilize the developments in [10] to express v_1 and $\sum w_{n1}$ as

$$v_1 = \max \left\{ 0, \left(\sum_{i=1}^{I} d_i^1 x_i + D_1 \right) \right\}$$

$$\text{and} \quad \sum_n w_{n1} = \max \left\{ 0, \left(- \sum_{i=1}^{I} d_i^1 x_i - D_1 \right) \right\} .$$

If we then associate the relevant interest charges with these expressions, we may transfer them to the functional where the expected value operator will reduce them to deterministic expressions. We may then treat constraints (5b) in an analogous manner and thus arrive at a new problem, which is a deterministic equivalent of (5).

We will first develop a deterministic equivalent for the situation in which borrowing at a fixed rate is possible, but in which lending is not possible. The amount of borrowing in any period will be given by

$$(12) \quad w_j = \max \left\{ 0, \left(- \sum_{i=1}^{I} d_i^j x_i - D_j + (1+r) w_{j-1} - v_{j-1} \right) \right\}$$

where the d_i^j and D_j are stochastic variables. w_{j-1} and v_{j-1}, of course, can be expressed in terms of the d_i^{j-1}, D_{j-1}, w_{j-2}, and v_{j-2}, and the expression for w_j may be developed recursively, since we may assume w_o (the initial outstanding borrowing) and v_o (the initial stock of surplus cash) to be known. Since, as we observed earlier, borrowing will not take place unless the cash flows require it, $w_j = 0$ when the quantity

$$\left(- \sum_{i=1}^{I} d_i^j x_i - D_j + (1+r) w_{j-1} - v_{j-1} \right)$$

is negative. We may thus again utilize the developments in [10] to express borrowing in a given period as

$$(13) \quad w_j = \tfrac{1}{2} \left(- \sum_{i=1}^{I} d_i^j x_i - D_j + (1+r) w_{j-1} - v_{j-1} \right) + \tfrac{1}{2} \left| - \sum_{i=1}^{I} d_i^j x_i - D_j + (1+r) w_{j-1} - v_{j-1} \right|$$

where the vertical lines indicate an absolute value.

Expressing w_{j-1} and v_{j-1} in terms of the recursive relationship defined by (13), we obtain

$$\begin{aligned} w_j = \tfrac{1}{2} \Bigg\{ & - \sum_i d_i^j x_i - D_j + (1+r) \tfrac{1}{2} \Bigg[\Big(- \sum_i d_i^{j-1} x_i - D_{j-1} + (1+r) w_{j-2} \\ & - v_{j-2} \Big) + \Big| - \sum_i d_i^{j-1} x_i - D_{j-1} + (1+r) w_{j-2} - v_{j-2} \Big| \Bigg] - \\ & - \tfrac{1}{2} \Bigg[\Big(\sum_i d_i^{j-1} x_i - D_j^{j-1} - (1+r) w_{j-2} + v_{j-2} \Big) + \Big| \sum_i d_i^{j-1} x_i + D_{j-1} \\ & - (1+r) w_{j-2} + v_{j-2} \Big| \Bigg] \Bigg\} + \tfrac{1}{2} \Bigg| - \sum_i d_i^j x_i - D_j + (1+r) \tfrac{1}{2} \Bigg[\Big(- \sum_j d_i^{j-1} x_i \end{aligned}$$

21a/ This involves a generalization of the theorems in [10] in that (a) the coefficients are random variables and (b) an extension of "2-stage" LPU[2] is involved here. (We previously also utilized this same generalization in [4] but failed to note it explicitly at that time.)

$$- D_{j-1} + (1+r)w_{j-2} - v_{j-2}) + \left| -\sum_i d_i^{j-1} x_i - D_{j-1} + (1+r)w_{j-2} - v_{j-2} \right| \Big]$$

$$-\tfrac{1}{2}\Big[\big(\sum_i d_i^{j-1} x_i + D_{j-1} - (1+r)w_{j-2} + v_{j-2}\big) +$$

$$+ \left| \sum_i d_i^{j-1} x_i + D_{j-1} - (1+r)w_{j-2} + v_{j-2} \right| \Big] \Big|$$

We may simplify this expression by combining terms as follows

$$w_j = \tfrac{1}{2}\Big\{ r/2 \left| \sum_i d_i^{j-1} x_i + D_{j-1} - (1+r)\, w_{j-2} + v_{j-2} \right| +$$

$$\left(\frac{2+r}{2}\right) \big(-\sum_i d_i^{j-1} x_i - D_{j-1} + (1+r)\, w_{j-2} - v_{j-2}\big) - \sum_i d_i^j x_i - D_j \Big\}$$

$$+\tfrac{1}{2} \Big| r/2 \left| \sum_i d_i^{j-1} x_i + D_{j-1} - (1+r)\, w_{j-2} + v_{j-2} \right| +$$

$$\left(\frac{2+r}{2}\right) \big(-\sum_i d_i^{j-1} x_i - D_{j-1} + (1+r)\, w_{j-2} - v_{j-2}\big) - \sum_i d_i^j x_i - D_j \Big|$$

To achieve further simplification, we observe that the interest term contributes at most a small fraction of w_j. Consequently, we may approximate w_j by[22/]

$$w_j \cong \tfrac{1}{2} \big(-\sum_i d_i^j x_i - D_j - \sum_i d_i^{j-1} x_i - D_{j-1} + w_{j-2} - v_{j-2}\big)$$

$$+\tfrac{1}{2} \left| -\sum_i d_i^j x_i - D_j - \sum_i d_i^{j-1} x_i - D_{j-1} + w_{j-2} - v_{j-2} \right|$$

Expressing w_{j-2} and v_{j-2} in terms of the recursive relationship (13) introduces the d_i^{j-2} and D_{j-2} into the expression for w_j. Following this procedure recursively to the initial period, we obtain

(14) $$w_j \cong \tfrac{1}{2} \big(-\sum_i \sum_i d_i^j x_i - D_j + w_o - v_o \big) +$$

$$\tfrac{1}{2} \left| - \sum_i \sum_i d_i^j x_i - D_j + w_o - v_o \right|$$

In other words, if the sum of the total net cash flows from new projects and the existing operations is a negative quantity (i.e., a net outflow) then the firm must borrow an equivalent amount of cash. If the sum of the flows is positive (i.e., a net inflow) then borrowing will not take place.

22/ This approximation eliminates the "generalized hypermedian" terms. See, e.g., A. Charnes, W. W. Cooper, G. L. Thompson [10] for a discussion of such hypermedians.

Reflection on (14) will reveal that borrowings expressed in this form are nothing other than a liquidity constraint, where the minimum liquidity level is identical to initial cash holdings (i.e., $L_j = M_o$, for all j). Consequently, we may transfer the expression for borrowings to the functional in a fashion analogous to the one we formulated for [4]. We may express the interest cost for a given year as

$$\frac{\bar{r}}{2}\left(-\sum_j \sum_i d_i^j x_i - \sum_i D_j + w_o - v_o\right) +$$
$$\frac{\bar{r}}{2}\left|\sum_j \sum_i d_i^j x_i - \sum D_j + w_o - v_o\right|$$

where $\bar{r}$ is the average rate paid for that year. The total available cash at the end of the horizon will be approximately

$$(15)\quad \sum_{j=1}^{J} \sum_{i=1}^{I} d_i^j x_i + \sum_{j=1}^{J} D_j - \frac{r_1}{2} \sum_{j=1}^{J} \left[\left(-\sum_{t=1}^{j} \sum_{i=1}^{I} d_i^t x_i - \sum D_t + w_o - v_o\right) \right.$$
$$\left. + \left| -\sum_{t=1}^{j} \sum_{i=1}^{I} d_i^t x_i - \sum_{t=1}^{j} D_t + w_o - v_o \right| \right]$$

if borrowing takes place only at the lowest rate. That is, cash assets will be the net cash flow resulting from existing assets and new investments less interest charges paid out on borrowings. When borrowings must be obtained at successively higher rates, we must have an interest term for each rate. For example, if borrowings in excess of an amount K_1 must be made at rate $r_2 > r_1$ then the interest term in (15) becomes

$$(15.1)\quad -\frac{r_1}{2} \sum_j \left[\left(-\sum_t \sum_i d_i^t x_i - \sum_t D_t + w_o - v_o\right) + \left| -\sum_t \sum_i d_i^t x_i - \right.\right.$$
$$\left.\left. \sum_t D_t + w_o - v_o \right| \right]$$
$$- \frac{r_2 - r_1}{2} \sum_j \sum_t \sum_i d_i^t x_i - \sum_t D_t - K_1 + w_o - v_o) +$$
$$+ \left| \sum_t \sum_i d_i x_i - \sum_t D_t - K_1 + w_o - v_o \right| \Big]$$

The second part of this expression (15.1) equals 0 for all values of borrowing up to K_1 and represents the actual borrowings times the interest premium for all borrowings which exceed K_1. A similar expression will be required for each additional level of interest rates which apply.

We may now express the deterministic equivalent of a risk version of (5) (in which we define $r_o = 0$), viz.

$$\text{Maximize } E \left\{ \left(\sum_{i=1}^{I} \hat{d}_i x_i + \sum_j \sum_i d_i^j x_i + \sum D_j \right) \right.$$

$$- \sum_{n=1}^{N} \frac{r_n - r_{n-1}}{2} \sum_{j=1}^{J} \left[\left(- \sum_{t=1}^{j} \sum_{i=1}^{I} d_i^t x_i - \sum_{t=1}^{j} D_t - K_n + w_o - v_o \right) + \right.$$

$$\left. \left. + \left| - \sum_{t=1}^{j} \sum_i d_i^t x_i - \sum_{t=1}^{j} D_t - K_n + w_o - v_o \right| \right] \right\}$$

(16) Subject to

(a) $\sum_i P_i x_i + \sum_h Q_h y_h \leq (1- \alpha)$

(b) $0 \leq x_i \leq 1, \qquad i = 1,\dots, I$

(c) $\sum_{i \in M[h]} x_i - y_h \leq m_h - 1 \qquad h = 1,\dots, H$

(d) $-\sum_{i \in M[h]} x_i + m_h y_h \leq 0,$

(e) $y_h \geq 0$

We observe at this point that this is similar to a portion of the development in [4],[23] (with the omission of a horizon posture constraint), where we have 1) added the horizon values of physical assets (the $\hat{d}_i$) to our objective function, 2) represented the cash flows generated by operation of existing assets by a distinct variable (D_j), and 3) collapsed the initial investment (d_i^o) into the total first year cash flow (d_i^1).

23/ See model (8.3) in [4]

The payback constraint appears as (16a) in a form that is consistent with discrete distributions of cash flows rather than in the continuous form. Thus, we see that our basic model formulated for continuous variables in [4], although it was developed from a consideration of the basic elements of the investment problem, is in fact a logical extension of Weingartner's model to include multi-dimensional elements of risk.

We now turn to the development of the dual to (16). In order to accomplish this we must take the expectations of the absolute values of certain random variables of discrete distributions. Although the notation is complicated, we do obtain a linear programming equivalent to (16). We turn now in the following sub-sections to develop this equivalent.

5.1 The Assignment of Discrete Probabilities and Their Joint Distributions

The discrete random variable d_i^j is to be defined by the following notation.

$$(17) \qquad d_i^j = \left\{ d^j_{ik_i^j} \text{ with probability } P^j_{ik_i^j} \geq 0 \text{ for all } k_i^j \in \mathcal{K}_i^j \right\}.$$

The symbol k_i^j ranges over the finite index set $\mathcal{K}_i^j$ and in this notation

$$\sum_{k_i^j \in \mathcal{K}_i^j} P^j_{ik_i^j} = 1 \text{ for any i and j.}$$

Similarly the random variables D_j are given by:

$$(18) \quad D_j = \left\{ D_{js} \text{ with probability } P_{js} \geq 0 \text{ for all } s \in S^j \right\}.$$

Here S^j is an appropriate index set for D_j and we require $\sum_{s \in S^j} P_{js} = 1$ for each j.

For each j define the following cartesian products of index sets:

$$(19) \quad \bar{\mathcal{K}}_I^j = \prod_{\substack{1 \leq t \leq j \\ 1 \leq i \leq I}} \mathcal{K}_i^t$$

and

$$(20) \quad \bar{S}^j = \prod_{1 \leq t \leq j} S_t.$$

We shall denote elements in $\bar{\mathcal{K}}_I^j$ by $\bar{k}_I^j$ and elements in $\bar{S}^j$ by $\bar{s}^j$. Thus, associated with any index point $(\bar{k}_I^j, \bar{s}^j)$ is the outcome

$$(d^1_{1k^1_1}, \ldots, d^j_{1k^j_1}, d_{2k^1_2}, \ldots, d^j_{2k^j_2}, \ldots, d^1_{Ik^1_I}, \ldots, d^j_{Ik^j_I}, D_{1s_1}, \ldots, D_{js_j})$$

with probability

$$(21) \qquad \prod_{\substack{1 \le t \le j \\ 1 \le i \le I}} p^t_{ik^t_i} \quad \prod_{1 \le t \le j} p_{ts_t}$$

assuming, without loss of generality, statistical independence.

5.2 Evaluation of the Expected Value of the Absolute Value in Terms of Discrete Random Variables

The variable part (with respect to decision variables x_i, $1 \le i \le I$) of the objective function in (16) is the following:

$$(22) \qquad \sum_{i=1}^{I} \left[\hat{d}_i + \sum_{j=1}^{J} E d^j_i + \frac{r_N}{2} \sum_{j=1}^{J} \sum_{t=1}^{j} E d^t_i \right] x_i$$

$$- E \left\{ \sum_{n=1}^{N} \frac{r_n - r_{n-1}}{2} \sum_{j=1}^{J} \left| - \sum_{t=1}^{j} \sum_{i=1}^{I} d^t_i x_i - \sum_{t=1}^{j} D_t - K_n + w_o - v_o \right| \right\}$$

where we assume that the $\hat{d}_i$'s are constants.

For each n we immediately obtain the following expectations in terms of the discrete random variables:

$$(23) \qquad E \sum_{j=1}^{J} \left| - \sum_{t=1}^{j} \sum_{i=1}^{I} d^t_i x_i - \sum_{t=1}^{j} D_t - K_n + w_o - v_o \right| =$$

$$= \sum_{j=1}^{J} \left\{ \sum_{\substack{\tilde{k}^j_I \in \tilde{K}^j_I \\ \tilde{s}^j \in \tilde{S}^j}} \left| - \sum_{i=1}^{I} \sum_{t=1}^{j} d^t_{ik^t_i} x_i - \sum_{t=1}^{j} D_{ts_t} - K_n + w_o - v_o \right| \prod p^t_{ik_i} \prod p_{ts_t} \right\}$$

The right hand side of (23) is a sum of absolute values of linear terms with all probability "removed". Thus, this expression is of the form,

(24) $$\sum_a \left| \zeta_a^n \right|$$

for each n, summed over a large index set. Where ζ Generally represents terms such as those apparent within the absolute values of (23). Now the objective function equivalent of (22) incorporates (with appropriate additional factors) $-\sum_a \left| \zeta_a^n \right|$ in its maximizing form. Therefore we introduce a variable Z_n' unconstrained in sign, for each n, and require the form:

(25) $$-Z_n' + \sum_a \left| \zeta_a^n \right| \leq 0.$$

We are now in a position to give a linear inequality system of supporting hyperplanes for inequality (25) which takes the form:

(26) $$-Z_n' + \sum_a \pm \zeta_a^n \leq 0.$$

for all possible assignments of + and - signs to terms in the summand. Thus, if there are Q terms in summand (24), then there are 2^Q such assignments of + and -'s and hence 2^Q linear inequalities in the system (26).

Returning now to the particular sum (of the general form (24))of interest, namely (23), we see that there are precisely

(27) $$Q = \sum_{j=1}^{J} \left| \tilde{K}_I^j \right| \left| \tilde{S}^j \right|$$

summands where $\left| \tilde{K}_I^j \right|$ = the number of elements in $\tilde{K}_I^j$ and similarly for $\tilde{S}^j$. We shall denote these orderings of + and -'s by θ and index these functions by the variable u, $1 \leq u \leq Q$. Thus, following the general developments of (25) and (26) applied to (23) we obtain the linear inequality system for each n

(28) $$\sum_{j=1}^{J} \sum_{\substack{\tilde{k}_I^j \\ \tilde{s}^j}} \theta_u (\tilde{k}_I^j, \tilde{s}^j) \left[-\sum_{i=1}^{I} \sum_{t=1}^{j} d_{ik_i}^t t x_i - \sum D_{ts_t} - K_n + w_o - v_o \right] \Pi P_{ik_i}^t t \Pi P_{ts_t} - Z_n \leq 0$$

for $1 \leq u \leq Q$.

5.2.1 Further Simplifications: Signum Random Variables

It would appear that (28) actually involves expectations of random variables closely related to the d_i^j's and the D_j's via the assignment of + and - signs. This is precisely the case and we refer to these altered random variables as "signum" random variables. More precisely, recall that the signum functions Θ_u, $1 \leq u \leq Q$ define a functional assignment of + and -'s to all terms in the summand (23). When restricted in the natural way to each subset of summands determined by the index set, say, $(\breve{K}_I^j, \breve{s}^j)$ for any given j, each Θ_u determines an expectation for a new random variable that is derived from the old one by assigning the so-determined + and - signs to sample points according to the overall Θ_u.

Thus, let

$$\left[\sum_{t=1}^{j} d_i^t\right]_u \tag{29}$$

denote the new (signum) random term derived from $\sum_{t=1}^{j} d_i^t$ by assigning + and - signs to its sample points according to the overall assignment function Θ_n^u. Similarly we define the signum random variable

$$\left[\sum_{t=1}^{j} D_t\right]_u. \tag{30}$$

In terms of the signum random variables (28) becomes:

$$-\sum_{i=1}^{I}\left\{\sum_{j=1}^{J} E\left[\sum_{t=1}^{j} d_i^t\right]_u\right\} x_i - z_n' \leq \sum_{j=1}^{J} E\left[\sum_{t=1}^{j} D^t\right]_u + \delta_u \left[K_n - w_o + v_o \right] \tag{31}$$

$$\text{for } 1 \leq u \leq Q \qquad \text{and} \qquad 1 \leq n \leq N,$$

and where

$$\delta_u = \sum_{\substack{\breve{k}_I^j \\ \breve{s}^j}} \Theta_u(\breve{k}_I^j, \breve{s}^j)\, \pi\, p_{ik_i^t}^t\, \pi\, P_{ts_t}$$

5.3 Primal and Dual Linear Programming Problems

At this stage of development we present the linear programming equivalent of (16) and its dual. We find it convenient to introduce

$$Z_n = -Z_n'.$$

We then obtain the following.

PRIMAL

$$\text{Max} \sum_{i=1}^{I} \left\{ \hat{d}^i + \sum_{j=1}^{J} E d_i^j + \frac{r_N}{2} \sum_{j=1}^{J} \sum_{t=1}^{j} E d_i^t \right\} x_i + \sum_{n=1}^{N} \frac{r_n - r_{n-1}}{2} Z_n$$

subject

to

$$\text{(a)} \quad -\sum_{i=1}^{I} \left\{ \sum_{j=1}^{J} E \left[\sum_{t=1}^{j} d_i^t \right]_u \right\} x_i + Z_n \leq \sum_{j=1}^{J} E \left[\sum_{t=1}^{j} D_t \right] + \gamma_u \left[K_n - w_o + v_o \right] \quad ; \quad 1 \leq u \leq Q, \; 1 \leq n \leq N$$

$$\text{(32)} \quad \text{(b)} \quad \sum_{i=1}^{I} P_i x_i + \sum_h Q_h y_h \leq 1 - \alpha$$

$$\text{(c)} \quad 0 \leq x_i \leq 1 \quad ; \quad 1 \leq i \leq I$$

$$\text{(d)} \quad \sum_{i \in m[h]} x_i - y_h \leq m_h - 1 \quad ; \quad 1 \leq h \leq H$$

$$\text{(e)} \quad -\sum_{i \in m[h]} x_i + m_h y_h \leq 0 \quad ; \quad 1 \leq h \leq H$$

and $\quad y_h \geq 0$

DUAL

$$\text{Min} \sum_{u,n} \tau_{u,n} \left\{ \sum_{j=1}^{J} E \left[\sum_{t=1}^{j} D_t \right]_u + \gamma_u (K_n - w_o + v_o) \right\} + \omega(1-\alpha) + \sum_i \mu_i \sum_h (m_h - 1) \eta_h^+$$

subject

to

$$\text{(a)} \quad \sum_{u,n} \tau_{u,n} \left\{ -\sum_{j=1}^{J} E \left[\sum_{t=1}^{j} d_i^t \right]_u \right\} + \omega P_i + \mu_i + \sum_h (\eta_h^+ - \eta_h^-) \geq \hat{d}_i + \sum_{j=1}^{J} E d_i^j + \frac{r_N}{2} \sum_{j=1}^{J} \sum_{t=1}^{j} E d_i^t \quad ; \quad 1 \leq i \leq I$$

(33) (b) $$\omega q_h \quad - \eta_h^+ + m_h \eta_h^- \geq 0 \quad ; \quad 1 \leq h \leq H$$

(c) $$\sum_u \tau_{u,n} \quad = \frac{r_n - r_{n-1}}{2} \quad ; \quad 1 \leq n \leq N$$

(d) $$\tau_{u,n}, \omega, \mu_i, \eta_h^+, \eta_h^- \quad \geq 0.$$

5.4 Risk Interpretations and Equivalences of the Model

Observe that the dual variables $\tau_{u,n}$ are associated with the supporting hyperplane constraints and that these are derived as equivalences for the need to maintain liquidity. While Q is very large we would expect many of the inequalities (a) in (32) to be redundant for any particular optimal solution x_i^*, $1 \le i \le I$ and Z_n^*, $1 \le n \le N$. Observe also that $\sum_{u} \tau_{u,n} = \frac{r_n - r_{n-1}}{2}$ implying that the non-zero $\tau_{u,n}$'s, for each n, partition the average difference in borrowing at levels K_{n-1} and K_n. The non-zero $\tau_{u,n}$ delimit periods when borrowing may take place at lower or at higher levels. This is so because an assignment of + and -'s via θ_u permit or forbid borrowing via the term in brackets in (28) and its effect on the expectation of the absolute value in (23) and finally its equivalent impact on total debt outstanding in any period. At the time of actual implementation of an optimal solution x_i^*, $1 \le i \le I$, borrowing may only be permitted in certain periods as determined by the binding supporting hyperplane constraints, in particular the terms $-\sum_{j=1}^{J} E\left[\sum_{t=1}^{j} d_i^t\right]_u$ for each i and periods j.

Observe also that the dual variable ω is associated with the original payback constraint, and the dual variables η_h^+, η_h^- are associated with the "adjoined" constraints. The quantity:

$$\sum_{u,n} \tau_{u,n}^* \left\{ -\sum_{j=1}^{J} E \left[\sum_{t=1}^{j} d_i^t \right]_u \right\} + \omega^* P_i + \sum_h (\eta_h^{+*} - \eta_h^{-*})$$

represents a risk premium which must be met by the project in order for it to be adopted. Note that this quantity may be positive, negative, or zero. That is, the term "risk premium" is to be regarded as generic and may in fact represent a risk subsidy to a project.

We need to examine this in more detail in order to indicate the nature of the different kinds of risk involved and so we proceed to a term-by-term interpretation as follows:

(i) The $\sum_{u,n} \tau^*_{u,n} \left\{ - \sum_{j=1}^{J} E\left[\sum_{t=1}^{j} d_i^t \right]_u \right\}$ term reflects the entire selection of projects and admissible lines of credit in what periods and up to what amounts. This quantity may be positive, negative, or zero. This term is a function of the probability of borrowing. It arises directly from the need to maintain liquidity.

(a) If this term is positive, then borrowing is not required (as a tendency) and therefore a smaller risk premium is required.

(b) If the sum is negative, then there is a tendency for borrowing in restricted periods and therefore the premium is larger.

(ii) The $\omega^* P_i$ term represents the risk premium required of a project if every project is individually required to have at least a probability α of payback. It is independent of other projects.

(iii) The remaining terms consist of a positive and a negative term in the dual variables η_h^{+*} and η_h^{-*}. The values of η_h^{+*} and η_h^{-*} derive from the dual constraints (33b), which must hold as equalities when the associated y_h^* is non-zero in optimum solution. These η_h^{+*} and η_h^{-*} variables arise, respectively, from the primal constraints (32d) and (32e) which are lower and upper limits, respectively, on the y_h variables. The η^{+*} and η^{-*} represent then an increase (if $\sum_h \eta_h^{+*} > \sum_h \eta_h^{-*}$) or a decrease (if $\sum_h \eta_h^{+*} < \sum_h \eta_h^{-*}$) in the risk premium required of an individual project, and this increase or decrease arises through the effect of the variables y_h, which represent the interactions between projects.

Dual constraint (33b) must hold as an equality when any y_h is positive ($y_h > 0$ implies that some of the associated $x_i > 0$). This relationship allows us to examine more directly the effect of the Q_h coefficients. We want to show how these Q_h terms can be interpreted to indicate the characteristics of the projects with which they are associated.

These Q_h values, we note, are simply summations of joint and individual probability terms. Consider, e.g., a binomial coefficient (e.g., m = 2). The coefficient, Q_h is determined as 24/

$$Q_h = (P_{AB} - P_A - P_B)$$

If the non-payback probability of the two projects adopted jointly is less than the sum of the individual non-payback probabilities, Q_h will be negative. If the non-payback probability of the two projects adopted jointly is greater than the sum of the individual non-payback probabilities, Q_h will be positive.

If for a particular non-zero y_h^*, the associated $Q_h > 0$, then the risk premium required for acceptance of all projects in the set represented by the y_h is increased by the amount $(\eta_h^{+*} - \eta_h^{-*})$, a positive quantity.25/ In other words, the premium required, in terms of expected present value, for acceptance of a project increases if the non-payback probability as part of a group is large compared to its non-payback probability as an individual project.

As the value of Q_h decreases, the quantity $(\eta_h^{+*} - m_h \eta_h^{-*})$ and therefore the quantity $(\eta_h^{+*} - \eta_h^{-*})$, must likewise decrease, so that

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24/ The determination of n^{th} order coefficients is detailed in Appendix A

25/ For $y_r > 0$, (33b) must hold as an equality, which requires that $\omega^* Q_h = \eta_h^{+*} - m_h \eta_h^{-*}$. Since $\omega^* > 0$, and m is a positive integer, this means that $\eta_h^{+*} > \eta_h^{-*}$ or equivalently $(\eta_h^{+*} - \eta_h^{-*}) > 0$.

the risk premium required for acceptance of the projects in a particular set, h, is a decreasing function of Q_h. When Q_h is a sufficiently large negative value, the quantity $(\eta_h^{+*} - \eta_h^{-*})$ will become negative. In other words, the risk premium required for acceptance of a project decreases if the non-payback probability as part of a group is small compared to its non-payback probability as an individual project. These dual variables η_h^{+*} and η_h^{-*} are measures of the portfolio effects associated with the acceptance of any particular set of projects.

To simplify matters we may assume that dispersion or some related measure such as variance, coefficient of variation, etc., is used to represent risk. The existence of these portfolio effects follows directly from the fact that the dispersion measure of the distribution of the sum of a number of stochastically independent variables is generally not equal to the sum of the dispersion measures of the variables. Consequently, if the accepted portfolio consists of more than a single project, the probability that the sum of certain of these stochastic cash flows will not achieve a specified level cannot be measured as a linear sum of individual project attributes, but must take into consideration the entire set of accepted projects. This effect is precisely measured by the values of the dual variables, η_h^{+*} and η_h^{-*}. These η_h^{+*} and η_h^{-*}, then, represent the effects of the acceptance of each project on the hurdle which other projects must pass in order to be accepted.

Evidently, the portfolio effects involve interactions between different projects and their risk-return relations. Nevertheless, we have by our transformation separated out the specific effect of the selection of one project on the selection of all others. We may also note that the existence of these dual variables is in no way related to the degree of interdependency of specific projects. Thus we may impute these portfolio effects for a specific project selection even though it consists of a mixture of statistically independent and interdependent projects.

In general, when selecting a portfolio from a large group of projects, any individual project i will be related to a maximum of $(2^N - 1)$ non-zero values of y_h^* (where N is the total number of non-zero x_i^*) through the primal constraints (32d) and (32e). These y_h^* will in general have both positive and negative Q_h values associated with them. For each non-zero y_h^*, dual constraint (33b) must hold as an equality, requiring that, when ω^* is non-zero, η_h^{+*} or η_h^{-*}, or both, be non-zero.

Thus, when the risk (payback) constraint (32b) is binding, (which is implied by $\omega^* > 0$), each $y_h^* > 0$ will give rise to a $\eta_h^{+*} > 0$, a $\eta_h^{-*} > 0$ or both. These η_h^{+*} and η_h^{-*} then are additive terms which modify the risk premium required for project acceptance on the basis of the project's contribution to (or detraction from) the desirability of the portfolio in terms of risk.

Since we do not know a priori which constraint terms will be non-zero at the optimum, we have no way of determining the net portfolio effect on a particular project prior to finding the optimum solution. However, there are characteristics of projects which will tend to be displayed in these portfolio effect variables, and these have been explored in [3].

Relationship (33a) also implies that money is interest free in any year that outstanding debt is not expected. Thus, if borrowing is never required, i.e., all $\tau_{u,n} = 0$, then any project with net cash flows (plus horizon asset value) which equal the required risk premium would be desirable. We observe that the dual constraint associated with each x_i is (33a) where the individual terms in the last summation are zero for any j in which no outstanding debt is expected.

In order for a project to be accepted (33a) must hold as an equality, i.e.,

$$(35)\quad \sum_{u,n} \tau_{u,n}^{*}\left\{-\sum_{j=1}^{J} E\left[\sum_{t=1}^{j} d_i^t\right]_u\right. + \omega^* P_i + \mu_i^* + \sum_h (\eta_h^{+*} - \eta_h^{-*}) = \hat{d}_i + \sum_{j=1}^{J} E d_i^j + \frac{r_N}{2} \sum_{j=1}^{J} \sum_{t=1}^{j} E d_i^t .$$

This expression states clearly the criteria for project acceptance. The expected sum of cash flows during the planning period plus the value of the asset at the end of the period must be sufficient to cover the payback risk premium plus (the marginal interest rate times the expected accumulative project deficit) and less (the marginal interest rate times the expected project surplus) for any year in which the firm is in debt.

Thus, a project which had an expected accumulative cash surplus in a year when expected borrowing was high could be desirable even though its expected total net cash flow was a very small positive quantity or even negative. A project with net negative cash flow would never be adopted, of course, unless the inflows preceded the outflows, since otherwise the firm would be ahead by simply holding cash. Such a project is, in essence, a loan wherein the firm obtains funds when needed and repays a larger amount later.

If the firm can lend money, say at the rate r_L, we need only add another expression to the objective of (16), expressing the expected interest earned for each period. By the same arguments presented earlier, the expected loans outstanding during a period will be

$$v_j = E\left[\ \tfrac{1}{2}\left(\sum_j \sum_i d_i^j x_i + \sum_j D_j + w_o - v_o\right) + \right.$$
$$\left. + \ \tfrac{1}{2}\left|\sum_j \sum_i d_i^j x_i + \sum_j D_j + w_o - v_o\right|\ \right]$$

The interest on this is then given by

$$r_L \quad E\left[\ \tfrac{1}{2}\left(\sum_j \sum_i d_i^j x_i + \sum_j D_j + w_o - v_o\right) + \tfrac{1}{2}\left|\sum_j \sum_i d_i^j x_i + \sum_j D_j + w_o - v_o\right|\ \right]$$

This earned interest term will be non-zero only in periods during which the expected value of loans outstanding is positive. These periods, of course, are those in which the expected value of outstanding debt is zero, since the firm will not lend and borrow funds during the same period.

Thus, the functional under these conditions would contain either a non-zero loan or a non-zero debt term for each period, and the dual constraint (33a) would then have a non-zero interest term for each period.

The dual conditions for project acceptance will not require that the left hand side of (35) be larger to the extent that a project shows a negative cumulative cash flow during years when the firm as a whole shows a cumulative cash surplus. Thus, when there is no profitable use for surplus funds, the individual projects are not penalized for not generating them when the firm as a whole generates them.

Similarly, these same conditions permit acceptance of an individual project with a lower value (asdetermined by $\hat{d}_i + \sum_j E\ (d_i^j)$) if it shows a positive cumulative cash flow during years when the firm as a whole shows a cumulative cash surplus, thus allowing for the opportunity of lending these surplus funds at interest.

Extension of this model to allow for project adoption in any period during the total planning period leaves our essential conclusions unchanged. The criterion for project acceptance remains the same, except that a project rejected for adoption in the initial period may now be adopted in a later period. This means that there will be a dual constraint of the form (33a) for each period for each project, and a project which does not meet this criterion of (35) during the first period may meet it during a later period due to a reduction in the interest term

$$r_N \sum_{k=1}^{J} \sum_{t=1}^{k} E\ (d_i^t)$$

If we consider project acceptance in any period, the cumulative cash flow of a project in the periods prior to its initiation is zero, so that the applicable interest penalties (or subsidies) are those starting with the period of project initiation. If the availability of very desirable projects causes a large outstanding debt during early periods, this may lead to the rejection of some projects in early periods and their acceptance in later periods when the interest term of (35) is smaller in magnitude.

6. The Integer Requirements

To the extent that an ordinary linear programming solution to our risk model contains fractional values for some of the x_{ij}^*, the solution is not wholly meaningful insofar as the related projects must be accepted or rejected on toto. In general it should be expected that some fractional values will be present. In order to determine a usable solution, a number of possible courses of action exist.

One option is to adopt fully all projects which are fractionally accepted. This will generally require a violation of some of the original constraints imposed on the selection. However, the constraints the firm faces in the capital budgeting situation are typically of a policy nature, rather than being associated with rigidly limited resource supplies.[26] Consequently, it is frequently possible to relax them, and adopt a program which is non-feasible with the original constraint limitations. The relevant management question is then to determine the possible gains to be achieved by a relaxation, and for this information we may look to the dual variables associated with the real constraints of our problem.

26/ See for instance [16], [29], and [33]. It is assumed that these "round'off" approaches do not cause really huge alterations in the budgets.

We know from the duality theory of linear programming that the dual variables associated with these real constraints are the shadow prices of the constraints. A firm may therefore determine the potential profitability of each possible constraint relaxation and decide accordingly whether such a relaxation is in order.

If such a relaxation does not provide an acceptable integer solution, we may turn to integer programming methods 27/ to achieve a final solution. This means that, in general, some of the fractionally accepted projects will be fully accepted, while others will be rejected.

This latter conclusion means that the relationship defined by (35) is not sufficient to discriminate between accepted and rejected projects. This implies that a system of subsidies and/or penalties must be imposed on the projects in order to have an unambiguous division between the accepted and the rejected projects. In Appendix B we attempt to determine more specifically the nature of these subsidies and penalties by reference to recent theoretical work by Egon Balas [1] in the duality relations of integer programs.

But we now note that one conclusion which can be drawn from Balas' [1] work is that for the normal capital budgeting problem only penalties will exist. We note that since the feasible space for the linear programming problem includes at least all points in the feasible space of the original problem, the value of the functional in the integer solution will be no greater than its value in the fractional solution.

The dual conditions for project acceptance, as given by (35), indicate that any project which meets these conditions is acceptable for inclusion in

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27/ For example Gormory's Cutting Plane approach, Balas' algorithm, etc.

a chosen portfolio. If (35) holds as a strict inequality, the project is in fact so desirable that it would be profitable to adopt more of the same type of project, if this were possible. If (35) holds as an equality, it means that a project with this level of expected flows is marginally acceptable, and the x_{ij} value for this project may be fractional in the optimal solution due to funds for its adoption being limited by a budget constraint or pay-back constraint. The conditions for acceptance defined by (35) are unaffected by the integrality requirement. If a project does not satisfy this criterion it cannot be profitably included in any portfolio. However, some projects which meet this criterion will have to be rejected. In terms of Balas' [1] integer duality theory, a penalty must be applied to these projects. This penalty would then appear as a positive term on the right hand side of (33a) and, consequently on the left hand side of (35), so that these projects would then no longer satisfy the acceptability criterion.

7. Implications.

The model of capital budgeting under risk presented here leads us to conclude that many traditional methods for dealing with risk, such as increasing the discount rate on individual projects, can lead to less than optimum investment selections. A fortiori this is likely to be the case when portfolios are involved since, then, the premiums on all other projects must be considered before establishing the net adjustment amounts on each project under consideration.

Our linear programming model brings out quite clearly the risk premiums required of individual projects when constraints are imposed to limit the risks to be admitted. It also indicates that these premiums should frequently be adjusted, either higher or lower, based on the contribution of the individual projects as parts of a portfolio. The fact that the dual linear programming problem gives us a direct additive formulation of the conditions required for acceptance of an individual project does not justify effecting such risk adjustments on a one-at-a-time basis relative to some assumed (or bogey) rate of interest, even when the conditions for integrality are waived. As we have seen, the portfolio effect can constitute a substantial part of the "hurdle" a project must pass for acceptance and this, in turn, depends on the <u>optimum</u> portfolio composition (or mix) allowed by the constraints. In fact, this adjustment may be <u>negative</u> and this <u>negative</u> adjustment value may even be large enough to make a project desirable even though it has a negative total expected cash flow.

This model has other useful features, particularly in the area of determining possible trade-offs. The value of ω^*, for instance, provides a measure of the extent to which the value of the functional might be improved in return for a relaxation in the payback constraint. Similar dual evaluators will exist for other chance constraints which may be imposed, and again these will offer direct access to the profit which is sacrificed in order to constrain the risks to the specified levels. Since the model presented here allows for cash flows to be described by arbitrary discrete distributions, it would be appropriate for situations where the estimated probability distributions of cash flows were appreciably asymmetric, multi-modal, etc.

The broader implications of this model for decentralized decision making, as well as the indicated relationship of project interdependence to the portfolio effect area discused in some detail in [3].

APPENDIX A

Computation of Probabilities

If, in a particular solution, the highest order non-zero term is of order n, we can define the decision variables in that term as the set N. Then all terms of lower order than n in which the decision variables are subsets of N will also be non-zero. There will be n terms of order (n-1), n(n-1)/2 terms of order (n-2), etc., to n terms of order 1. This means that a function equal to the sum of the various one-project, two-project, three-project, etc. probabilities will be double-counting the probabilities. With the probabilities expressed as we have noted, the value of the highest order non-zero term represents the total probability of achieving the specified level. However, all the lower order terms containing x_i only of accepted projects are also non-zero. Consequently, to have a function represent the true probability, each term must subtract the scalar value of the next lower order terms from the total value of the function.

We now proceed to develop this expression analytically by first extending our definition of P as follows. Let $N = (1,2, \ldots, I)$, the set of all projects and $A \subseteq N$, that is A is a collection of projects. Let

P_A = the probability of achieving a negative cash flow in T periods if only projects in A are initiated and all other projects (N-A) are rejected. When $A = \emptyset$, the empty set, $P_\emptyset$ = the probability of achieving a negative

We now simplify the expression of p(x) inductively term by term starting with n = 1. The I first order terms in the expression are $\sum_i P_i x_{ij}$. If any one project, say A, accepted, the result is $x_A = 1$ and the expression becomes equal to P_A. The (I(I-1)/2) second order terms are

$$\sum_{\substack{i,m \\ i-m}} (P_{im} - P_i - P_m) x_{ij} x_{mj}$$

where the summation is over subscripts, i and m. Note, we are still requiring integer values only, $0 \leq x_i \leq 1$. Hence, if any two projects, A and B, are accepted, three terms will be non-zero, and the P_A, P_B values in the second order term will exactly cancel the two non-zero, first order terms. Similarly, the (I(I-1) (I-2)/6) third order terms must be:

$$\sum_{i,m,m} (P_{imn} - P_{im} - P_{in} - P_{mn} + P_i + P_m + P_n) x_{ij} x_{mj} x_{nj}$$

If any three projects are accepted, then three second order terms will be non-zero and the corresponding probabilities must be cancelled. However, the sum of the three second order terms contains each of the single-project probabilities <u>twice</u>, and so these must be added to the third order terms. The higher order terms are formed in a similar manner, subtracting and adding the lower order probabilities in order form n-1, n-2, etc., to 1.

The general form of these expressions is then:

$$\left[\sum_{A \subseteq R_o} P_A \, (-1)^{n(A)} \, (-1)^{n(R_o)}\right] \prod_{i \in R_o} x_i$$

where n(A) = the number of elements of A and where $R_o \subseteq I$ ranges over all possible subsets of projects under consideration. Since $P_\emptyset = 0$, this term need never appear in the problem. We shall define

$$Q_{R_o} = \sum_{A \subseteq R} P_A \, (-1)^{n(A)} \, (-1)^{n(R_o)}$$, <u>for example</u>,

cash flow in T periods if no projects are accepted = 0.

Let $P_{A,\bar{i}}$, for i not in A, be defined as $P_{A,\bar{i}} = P_A - P_{A,i}$

Similarly $P_{A,\bar{i},\bar{j}} = P_A - P_{A,i} - P_{A,j} + P_{A,i,j}$ when i and j are both not in A. In this manner we obtain the general definition of $P_{A,\bar{B}}$ when $A \cap B = \emptyset$, that is A and B are disjoint subsets of N.

The following lemma, whose proof we omit, follows from our definition of the "-" notation

Lemma 1. Let Y be any subset of N.

Then $\sum_{A \subseteq Y} P_{A,\,\overline{Y-A}} = 0.$

If B is disjoint from Y, then $\sum_{A \subseteq Y} P_{B,A,\,\overline{Y-A}} = P_B.$

For each selection of the decision vector $x = (x_1, x_2, \ldots, x_n)$, where $x_i = 1$ if project i is accepted and $x_i = 0$ if project i is rejected, we define $p(x)$ to be the probability of achieving a negative cash flow in T periods given x.

Proposition 1 $p(x)$ as defined above is given by:

$$\sum_{A \subseteq N} P_{A,\,\overline{N-A}} \left[\prod_{j \in N-A} (1-x_j)\right],$$ where by convention $\prod_{j \in \emptyset} (1-x) = 1.$

Proof. Given $A_o \subseteq N$, $A_o \neq \emptyset$, set $x_i^o = 1$ if $i \in A_o$ and $x_i^o = 0$ if $i \notin A_o$. Then we must show that $p(x^o) = P_{A_o}$. To do this, first observe that

$\prod_{j \in N-A} (1-x_j) \neq 0 \Longleftrightarrow N-A \subseteq N-A_o \Longleftrightarrow A \supseteq A_o$. Since $A_o \neq \emptyset$, $\prod_{j \in N} (1-x_j) = 0$,

and it follows that $p(x^o) = \sum_{A \supseteq A_o} P_{A,\overline{N-A}} = \sum_{A \supseteq A_o} P_{A_o,\, A-A_o,\, \overline{N-A}}.$

But as A ranges over subsets containing A_o $(A \supseteq A_o)$, $A-A_o$ ranges over all subsets of $N-A_o$, as does $N-A$.

Therefore $p(x^o) = \sum_{Q \subseteq N-A_o} P_{A_o,\ Q,\ \overline{(N-A_o) - Q}}$ and applying Lemma 1,

with $y = N-A_o$ yields $p(x^o) = P_{A_o}$. The only remaining case occurs with

$A_o = \emptyset$. In this case $\prod_{j \in N} (1-x_j) = 1$ and $p(x^o) = \sum_{A \subseteq N} P_{A,\overline{N-A}} = 0$,

which is what is required, Q. E. D.

This gives

$$Q_{im} = (P_{im} - P_i - P_m)$$

$$Q_{imn} = (P_{imn} - P_{im} - P_{in} - P_{mn} + P_i + P_m + P_n)$$

and so on.

The construction of a probability function for a large group of projects then requires first the computation of the probability of each of the possible mutually exclusive outcomes which meets the specified level of cash flow (e.g., for payback). But the determination of the probabilities associated with each pair, triad, etc., of projects is less difficult than it appears. If the various individual project cash flows are ordered on the basis of scalar value within each period and each project, the joint probabilities can be determined by starting at the maximum (or minimum) values of cash flows and continuing in order until the cash flow value (D_s) does not meet (or first achieves) the specified level.

In practical applications this should result in substantially less computational effort than would be required to determine the probabilities of all possible outcomes. To put this differently, this simplification arises as a reflection of our concern with only one side of the probability distribution of outcomes.[28/]

28/ E.g., as distinguished from other approaches which utilize such measures as the variance or coefficient of variation in outcomes, etc. See e.g., the discussion of the concept of semi-variance in [20], pp. 188-201.

Consider the data presented below. We wish to determine the probability of achieving payback in one year for all possible decisions. We do this by determining the investment required for each possible decision, then determining the cash flow combinations which equal or exceed the investment, and finally determining the probability of each of these cash flow combinations.

The estimates of cash flow, with associated probabilities of occurrence, are

	Project 1		Project 2		Project 3	
	Cash flow	Probability	Cash flow	Probability	Cash flow	Probability
	d^i_{s1}	p^k_{s1}	d^k_{s2}	p^k_{s2}	d^k_{s3}	p^k_{s3}
1st year	2	.3	3	.4	3	.4
	3	.5	5	.5	5	.5
	5	.2	7	.1	7	.1
2nd year	2	.5	2	.5	3	.4
	4	.4	5	.4	5	.5
	5	.1	6	.1	7	.1
3rd year	1	.3	1	.4	2	.3
	3	.6	3	.5	3	.4
	6	.1	5	.1	4	.3
Cost	$d^o_1 = -6$		$d^o_2 = -5$		$d^o_3 = -7$	

We assume T = 1. We then develop a risk constraint by requiring payback within one year with a probability of at least $1-\alpha$.

The results for the example problem are tabulated below.

(1) Decision (projects adopted)	(2) Investment	(3) Cash flow combinations which achieve payback	(4) Probability of (3)	(5) Total Prob.	(6) P=1-(5)
1,2,3	18	$d^1_{31} + d^1_{32} + d^1_{33}$	(.2)(.1)(.1)	.002	.998
1,2	11	$d^1_{31} + d^1_{32}$	(.1)(.2)	.02	.98
1,3	13	none	0	0	1.0
2,3	12	$d^1_{32} + d^1_{33}$	(.1)(.1)		
		$d^1_{32} + d^1_{23}$	(.1)(.5)		
		$d^1_{22} + d^1_{33}$	(.5)(.1)	.11	.89
1	6	none	0	0	1.0
2	5	d^1_{22}	.5		
		d^1_{32}	.1	.6	.4
3	7	d^1_{33}	.1	.1	.9

Note that in computing such a measure we need not evaluate exhaustively all possible combinations of outcomes. In this example, there are a total of 64[29] different mutually exclusive outcomes, yet we need compute the outcome and probability of only ten of these. As an example of the computational routine consider the decision to adopt all three projects. The outflow for this decision is 18. We start with the highest positive level of cash flow estimated for each project, $d_{31}^{1} + d_{32}^{1} + d_{33}^{1}$. We know that if this combination does not achieve our specified level, then no other combination of flows will. We find that $d_{31}^{1} + d_{32}^{1} + d_{33}^{1} = 19$, since $19 - 18 = 1$, we know that if the next lower level, any combination containing it will not achieve the specified condition. By inspection this proves to be the case.

The same procedure applies to the various possible combinations of two adopted projects and one adopted project. With no project adopted, the probability of achieving payback is, of course, 1.

We observe that with this approach, the probabilities of all possible outcomes after a decision is made must sum to 1. Therefore, in determining the probability of achieving a specified level, we may compute either the probability of achieving it or the probability of not achieving it. Since these two events include all possible outcomes and are mutually exclusive, their sum is 1. Consequently, we need to compute at most the probabilities of one-half the total number of possible outcomes to determine the appropriate coefficients for a chance constraint of this type.

29/ The sum of 27 possible outcomes if three projects are adopted, 27 if two are adopted, 9 if one is adopted, and 1 outcome if no project is adopted.

APPENDIX B

Duality Relations of Integer Programs

The interpretation of a subsidy-penalty system in programs such as our formulation of the capital budgeting problem obtains via Egon Balas work in [1]. Balas provides a dual to the mixed integer linear program

$$
\begin{array}{lll}
\text{Maximize} & cx & \\
\text{Subject to} & Ax \leq b & \\
 & x_j \geq 0, & j = 1, \dots, N \\
 & x_j \text{ integer}, & j \in N_1,\ N_1 \leq N
\end{array}
$$

as

$$
\begin{array}{lll}
\max\limits_{x}\ \min\limits_{u} & ub - v^1 x^1 & \\
\text{Subject to} & uA - v = c & \\
 & u, x \geq 0 & \\
 & x_j \text{ integer}, & j \in N_1
\end{array}
$$

$$
v_j \begin{cases} \text{unconstrained}, & j \in N_1 \\ \geq 0, & j \in N_1 \end{cases}
$$

Whereas in the ordinary linear programming problem we are looking for a feasible solution to the primal with the property that the associated solution to the dual is also feasible, we may observe that in the integer problem that the dual comes "as close as possible" to satisfying the dual constraints. The dual slack variables, v_j corresponding to an integer constrained primal variable are unconstrained. Therefore, the dual constraints may be "violated" (in the normal sense of a linear program) when $v_j < 0$. This degree of violation, however, appears in the dual functional, so that the value of the functional is "adjusted" to correspond to the "gap" between the non-integer optimum and the integer optimum.

This gives rise to a generalized shadow price system consisting of non-negative prices u_i associated with each constraint i, and subsidies or penalties v_j associated with each integer variable j. The actual values of these subsidies or penalties (which we shall denote as "Balas type subsidies") can be determined from the integer solution to the linear program. Balas shows further[30] that for a model such as ours, wherein there are no requirements of discreteness on the dual variables, that for the vector x of integer variables

$$(uA - c - v)\, x = 0$$

where v is the vector of subsidies and penalties assigned to the integer variables.

For our model (16), this relationship takes on the specific form

$$(36) \quad \Big[\lambda P_i + \sum_h (\eta^+_h - \eta^-_h) + \mu_i - d_i - \sum_j E(d_i^j) - r_N \sum_{j=1}^{J} \sum_{t=1}^{j} E(d_i^t) - v_i \Big] x_i = 0$$

The capital budgeting problem is a special case of discrete programming, since the variables are permitted to take on only the discrete values of 0 and 1. The unity upper bounds restrict those variables for which the ordinary fractional solution would be $x_i \geq 1$, so that in order to achieve an integer solution we need only adjust certain fractional variables to a value of 1 or 0.

If we are requiring that the integer solution be primal feasible, then the optimal integer solution must have a smaller objective than the optimal fractional solution by the amount $v^1 x^1$. These v_j values will be associated with x_i which must be reduced below their optimum fractional values (i.e., to 1), (36) is satisfied with $v_i = 0$, since (33a) holds as an equality at the

---

30/ [1], p. 25-26.

fractional optimum.

We may observe that any $v_i < 0$ will be sufficient to require $x_i = 0$ if (35) is fulfilled as an equality in the fractional solution. However, to determine the actual values of v_i which will achieve true duality, we must first find the optimal integer solution.

In the capital budgeting case, then, rather than a mixture of penalties and subsidies, the v_j will in general represent only penalties. The existence of such penalties is a substantial block to decentralized decision-making in the capital budgeting area, since these penalties are a result of the optimization, and are not available prior to determination of the optimum program.

These penalties are created by the integer requirements, and will, in general, bear no relationship to the penalties and subsidies of our previous development represented by the dual variables η^{+*} and η^{-*}, which arise from completely independent sources.

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13. ABSTRACT

Chance Constrained (C^2) Programming and Linear Programming under Uncertainty (LPU^2) are joined together in order to deal with different risks and uncertainties which are commonly encountered in capital budgeting. This includes payback period protection via chance constraints formulated to cover (or bound) a possible loss of future opportunities during the payback period. It also includes liquidity requirements formulated preemptively via LPU^2 to provide protection against possible cash (or liquidity) shortages at specified times.

The case of arbitrary discrete distributions is examined and new formulations are developed which model economic, statistical, and technological decision interdependencies. Relations to geometric programming are indicated prior to reducing these formulations to zero-one integer programming (deterministic) equivalents. Duality relations obtained from these formulations provide separate evaluators for yield, risk, portfolio and liquidity effects of cash investment. Finally, relations to "Balas-type" subsidy and penalty adjustments are noted.

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